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ON SOME PROPERTIES OF GENERAL MANIFOLDS RELATING TO EINSTEIN'S THEORY OF GRAVITATION.

BY J. A. SCHOUTEN AND D. J. STRUIK.

In two very interesting communications in this JOURNAL* Mr. E. Kasner recently deduced some properties of the four-dimensional manifolds for which $G_{\mu\nu} = 0$. Now it is possible to demonstrate two of these properties in a very short way for n-dimensional manifolds without calculating any three-index-symbol, as a direct consequence of some general theorems. Further it is very easy to give a generalization for manifolds, which obey not Einstein's gravitational equations but the more general equations of de Sitter for a quasi-spherical world.

By V_n is meant a general manifold of n dimensions, by R_n a euclidean one, by S_n a manifold with constant Riemannian curvature and by C_n a manifold which is conformally representable on an R_n.

1. In a recent publication† one of us showed that the necessary and sufficient condition for $n > 3$, that a V_n is a C_n, is that the curvature-tensor (Riemann-Christoffel-Tensor) $B_{\rho\mu\nu\sigma}$ can be written in the form:

$$B_{\rho\mu\nu\sigma} = \frac{1}{n-2}(g_{\rho\nu}L_{\mu\sigma} - g_{\mu\nu}L_{\rho\sigma} - g_{\rho\sigma}L_{\mu\nu} + g_{\mu\sigma}L_{\rho\nu}); \tag{1}$$

$$\rho, \mu, \nu, \sigma = a_1, \cdots, a_n$$

where $L_{\mu\sigma}$ is any symmetrical tensor. From this we conclude

$$G_{\mu\nu} = -L_{\mu\nu} - \frac{1}{n-2}L_{\rho\sigma}g^{\rho\sigma}, \tag{2}$$

and

$$L_{\mu\nu} = -G_{\mu\nu} + \frac{1}{2(n-1)}G_{\rho\sigma}g^{\rho\sigma}. \tag{3}$$

Hence $B_{\rho\mu\nu\sigma}$ vanishes for a C_n if $G_{\mu\nu}$ vanishes:

Of all C_n the only ones for which $G_{\mu\nu} = 0$ are R_n.

For $n = 3$ $B_{\rho\mu\nu\sigma}$ has always the form (1), hence:

Every V_3 for which $G_{\mu\nu} = 0$ is an R_3.

This theorem can be generalized in the following way. If

$$G_{\mu\nu} = \lambda g_{\mu\nu} \tag{4}$$

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* AM. JOURNAL OF MATHEMATICS, 43 (1921).

† J. A. Schouten, "Ueber die konforme Abbildung n-dimensionaler Mannigfaltigkeiten mit kwadratischer Maszbestimmung auf eine Mannigfaltigkeit mit euklidischer Maszbestimmung," *Math. Zeitschr.*, 1921, pp. 58–88.

(in relativity the equation of de Sitter for a quasi-spherical world) from (3) follows

$$(5) \qquad L_{\mu\nu} = -\frac{n-2}{2(n-1)}\lambda g_{\mu\nu},$$

and thus:

$$(6) \qquad \begin{aligned} B_{\rho\mu\nu\sigma} &= -\frac{\lambda}{2(n-1)}(g_{\rho\nu}g_{\mu\sigma} - g_{\mu\nu}g_{\rho\sigma} - g_{\rho\sigma}g_{\mu\nu} + g_{\mu\sigma}g_{\rho\nu}) \\ &= -\frac{\lambda}{n-1}(g_{\rho\nu}g_{\mu\sigma} - g_{\mu\nu}g_{\rho\sigma}). \end{aligned}$$

Now it is a well-known theorem that $B_{\rho\mu\nu\sigma}$ being of this form the V_n is an S_n:

Of all C_n the only ones for which $G_{\mu\nu} = \lambda g_{\mu\nu}$ are S_n.
Every V_3 for which $G_{\mu\nu} = \lambda g_{\mu\nu}$ is an S_3.

2. If a V_n is imbedded in a V_{n+1}, and if the indices μ, ν, ρ, σ belong to the fundamental variables in the first manifold, then the generalized theorem of Gauss is:

$$(7) \qquad B_{\rho\mu\nu\sigma}' = B_{\rho\mu\nu\sigma} - (h_{\rho\nu}h_{\mu\sigma} - h_{\mu\nu}h_{\rho\sigma}),$$

where $B_{\rho\mu\nu\sigma}$ is the curvature-tensor of the V_n, $B_{\rho\mu\nu\sigma}'$ the curvature-tensor of the V_{n+1} and $h_{\rho\nu}$ the (symmetrical) second fundamental tensor of the V_n with respect to the V_{n+1}.* Now if the V_{n+1} is an R_{n+1}, $B'_{\rho\mu\nu\sigma} = 0$ and from (7) we get:

$$(8) \qquad G_{\mu\nu} = g^{\rho\sigma}h_{\mu\rho}h_{\sigma\nu} - h_{\mu\nu}h,$$

where

$$(9) \qquad h = h_{\rho\sigma}g^{\rho\sigma}.$$

Now we introduce a system of rectangular coördinates in the principal directions of the symmetrical tensor $h_{\mu\nu}$, which are, according to (8), also the principal directions of $G_{\mu\nu}$. Let the orthogonal components be denoted by the indices $i, j = 1, \cdots, n$. Then $G_{\mu\nu}$ has n (real) components G_{ii}, and $h_{\mu\nu}$ has n (real) components h_{ii}, the components G_{ij}, h_{ij}, $i \neq j$ being zero, and (8) becomes

$$(10) \qquad G_{ii} = h_{ii}h_{ii} - h_{ii}\sum_{j}^{1,\ldots,n} h_{jj}$$

where no summations are to be taken that are not indicated by the sign Σ. If now $G_{\mu\nu} = 0$, we get from (10) that either all components h_{ii} vanish or all components h_{ii} but one, in both cases from (7) follows that $B_{\rho\mu\nu\sigma}$ vanishes. Thus:

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* See the above cited paper p. 24, or Bianchi-Lukat, Vorlesungenüber Differentialgeometrie, first edition, p. 623, or G. Ricci, Atti Acad. Lincei (1902), p. 359.

No V_n with $G_{\mu\nu} = 0$, that is not an R_n, can be regarded as imbedded in an R_{n+1}. The R_n is either flat or developable.

This theorem can also be generalized. If

$$G_{\mu\nu} = \lambda g_{\mu\nu}, \qquad G_{\mu\nu}' = \lambda' g_{\mu\nu}',^{*}$$

where $G_{\mu\nu}'$ belongs to the V_{n+1}, then (10) gets the form:

$$\lambda - \lambda' = h_{ii}h_{ii} - h_{ii} \sum_{j}^{1,\ldots,n} h_{jj}.$$

For $n = 2$ h_{11} and h_{22} are not determined by these equations. For $n \neq 2$ it is easy to prove that the only possible solutions are:

$$\begin{aligned} h_{11} = \cdots = h_{mm} &= \pm\sqrt{\frac{m'-1}{m-1}}\,(\lambda - \lambda'), \\ h_{m+1,\,m+1} = \cdots = h_{nn} &= \mp\sqrt{\frac{m-1}{m'-1}}\,(\lambda - \lambda'), \\ m' = n - m; \qquad & m = 0, 2, 4 \text{ for } n = 4, \\ & m = 0, 2, 3, \cdots, n-2, n \text{ for } n > 4. \end{aligned}$$

For these manifolds the generalized Codazzi-equations[†] give the following results:

If for a V_n in C_{n+1} $h_{11} = \cdots = h_{mm}$ is a complete group of equal principal orthogonal components of the second fundamental tensor, then the corresponding congruences of principal curvature form a V_m with only umbilical points with respect to the C_{n+1}. If the C_{n+1} is a S_{n+1} the value of $h_{11} = \cdots = h_{mm}$ on each of the $\infty^{n-m}V_m$ remains constant. If $h_{11} = \cdots = h_{mm}$ remains constant in the V_n, the $\infty^{n-m}V_m$ are geodesic in the V_n.

In consequence of these theorems and the above-mentioned solutions we can prove the theorem:

A V_n with $G_{\mu\nu} = \lambda g_{\mu\nu}$, that is imbedded in an S_{n+1}, $n > 3$, is either a manifold with only umbilical points (a hypersphere) or it contains $\infty^{n-m}V_m$ and also $\infty^{m}V_{n-m}$ with only umbilical points with respect to the S_{n+1}, where m may have the values $2, \cdots, n-2$. The V_m are geodesic with respect to the V_n and the value of the principal radii of curvature remains constant on the V_n.

Also it can be proved that each of the above-mentioned V_m is imbedded in an S_{m+1} that is geodesic in the S_{n+1} and each of the V_{n-m} in a geodesic S_{n-m+1}.

Hence a V_4 in S_5 with $G_{\mu\nu} = \lambda g_{\mu\nu}$ contains two systems of S_2 with only umbilical points, which have all the same curvature.

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[*] For a V_2 in S_3 these equations are identities.

[†] Compare Ricci, l.c.

3. The first theorem mentioned under 1 is a special case of a more general theorem.*

If a V_n with fundamental tensor $g_{\mu\nu}$ is represented conformally on a V_n with fundamental tensor $'g_{\mu\nu}$:

$$'g_{\mu\nu} = \sigma^{-1} g_{\mu\nu}, \tag{11}$$

then we can deduce for the curvature-tensors $B_{\rho\mu\nu\sigma}$ and $'B_{\rho\mu\nu\sigma}$ of both V_n:

$$'B_{\rho\mu\nu\sigma} = B_{\rho\mu\nu\sigma} + \tfrac{1}{4}(g_{\rho\nu}p_{\mu\sigma} - g_{\mu\nu}p_{\rho\sigma} - g_{\rho\sigma}p_{\mu\nu} + g_{\mu\sigma}p_{\rho\nu}), \tag{12}$$

where

$$p_{\mu\sigma} = 2s_{\mu\sigma} + s_\mu s_\sigma - \tfrac{1}{2}s_\rho s_\nu g^{\rho\nu} g_{\mu\sigma} \tag{13}$$

$$s_\mu = \frac{1}{\sigma}\frac{\partial\sigma}{\partial x^\mu}\,.$$

From (11) follows:

$$'G_{\mu\nu} = G_{\mu\nu} + \tfrac{1}{4}(2p_{\mu\nu} - g_{\mu\nu}p - np_{\mu\nu}). \tag{14}$$

Now when $G_{\mu\nu} = 0$ and $'G_{\mu\nu} = 0$ from (14) we get $p_{\mu\nu} = 0$, hence

$$'B_{\rho\mu\nu\sigma} = B_{\rho\mu\nu\sigma}. \tag{15}$$

When two V_n with $G_{\mu\nu} = 0$ are conformally representable on each other the Riemann-Christoffel curvature-tensors are equal.

DELFT, HOLLAND.

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[*] Schouten, ab. cit. p. 79.